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Analysis and computation of the vibration spectrum of the shallow circular cylindrical shell with rectangular domain

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Abstract

An asymptotic method of Bolotin, for the computation of eigenvalues of self-adjoint problems on rectangular domains, is extended to the shallow shell equations for the vibrating circular cylindrical shell. These same eigenfrequencies are then computed using the Legendre-tau spectral method. The asymptotic and numerical results are seen to be in good agreement and, as expected, approach those of the flat plate as the curvature tends to zero.

1. Introduction

Resonant eigenfrequency analysis is essential to the design and control of vibrating structures. In this paper we estimate the vibration spectrum of the equations for the shallow (and thin) circular cylindrical shell, on a rectangular domain and with strongly clamped boundary conditions. The equations modelling shell vibration constitute a system of a partial differential equations with order higher than those for membranes (order 2) and plates (order 4). Consequently, analysis of the spectrum is much more challenging.

Several authors have studied the vibration of cylindrical shells (see, e.g., Refs. [1–8]). However, these studies treat shells which are *closed* in the circumferential direction, allowing for the assumption of waveforms which automatically satisfy the conditions of smoothness in the circumferential variable. Further, as we explain in Section 2, some of these authors apply the *shallow* shell equations to these closed shells, despite the limitations implicit in the assumption of shallowness.

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In this work, we treat the shallow cylindrical shell without the assumption that it is closed. Our mathematical analysis is based on a method of Bolotin [9], which has been shown to be equivalent to the wave method of Keller and Rubinow [10] in the case of the vibrating Kirchhoff thin plate (see, Ref. [11]). We suspect that they are equivalent for shallow shells, as well. Therefore, it is our opinion that the level of mathematical rigor in this approach is quite high.

Even though we treat only the case where the shall is strongly clamped, our methods generalize to all combinations of energy-conserving boundary conditions (simply supported, roller supported and free) and, further, to more complicated models for the cylindrical shell. This paper provides benchmark data, both asymptotic and numerical, for comparison with future experimental results.

The outline of this paper is as follows: in Section 2 we provide a brief discussion of the model for the shallow cylindrical shell; in Section 3 we apply a generalization of Bolotin's asymptotic method to estimate the shell's vibration spectrum; in Section 4 we apply the Legendre-tau spectral method to this same problem; and in Section 5 we present benchmark data and a comparison of the results.

2. Model: the shallow shell equations for the circular cylindrical shell

The classical partial differential equations describing the deflection of a thin circular cylindrical shell (e.g., according to Timoshenko [12] and Donnell [13]), neglecting rotary inertia and shear deformation, are

$$
\begin{aligned}
&U_{xx} + \frac{1-v}{2} U_{yy} + \frac{1+v}{2} V_{xy} - \frac{v}{R} W_x = -\frac{(1-v^2)}{Eh} q_x, \\
&\frac{1+v}{2} U_{xy} + \frac{1-v}{2} V_{xx} + V_{yy} - \frac{1}{R} W_y = -\frac{(1-v^2)}{Eh} q_y, \\
&\frac{h^2}{12} \Delta^2 W + \frac{1}{R^2} W - \frac{1}{R} V_y - \frac{v}{R} U_x = \frac{1-v^2}{Eh} q_z.
\end{aligned} \tag{1}
$$

where x and y are the independent variables (y is the circumferential variable, while x varies along each generator); U, V and W are the x, y and z direction deflections, respectively; q_x, q_y and q_z, the loads in the positive x, y and z directions, respectively (with z increasing from "inside" to "outside" the shell); and the shell constants R, v, E and h are the shell's radius, Poisson ratio, modulus of elasticity and thickness. Finally,

$$
\Delta^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2
$$

is the biharmonic operator.

Now, in order to compute the shell's vibration spectrum, one must replace each load by the corresponding inertia term. Then, assuming (as we do throughout this paper) that the wave numbers are sufficiently large, we may neglect the u- and v-inertia terms (see Refs. [9,14]). A few authors [3,4,7] have used this model to calculate the vibration spectrum for the closed cylindrical shell. However, a very nice presentation of the equations of shallow shell theory can be found in Ref. [15], and it can be shown quite easily that this model leads to the partial differential equations

(1) in the case of the circular cylinder. Therefore, any study based on these equations must be considered subject to the limitations of the shallowness assumption. For example, Ref. [16] states that a cylindrical shell is shallow if ''its radius is greater by one order of magnitude than (its) linear dimensions.'' Thus, in some circumstances, it may not be appropriate to use PDEs (1) for closed cylindrical shells.

To be precise, then, let us give the dimensional version (with signs adjusted) of those equations in Ref. [15] which we need. In terms of the forces-per-unit-length N_{ij}, the moments-per-unit-length M_{ij}, the strains E_{ij}, the changes-of-curvature K_{ij}, and the displacements U, V and W, we have

equilibrium equations:

$$N_{11x} + N_{12y} = -q_x, \quad N_{12x} + N_{22y} = -q_y,$$
$$M_{11xx} + 2M_{12xy} + M_{22yy} + \frac{1}{R}N_{22} = -q_z; \tag{2}$$

strain–displacement relations:

$$E_{11} = U_x, \quad E_{22} = V_y - \frac{1}{R}W, \quad E_{12} = \tfrac{1}{2}(U_y + V_x),$$
$$K_{11} = -W_{xx}, \quad K_{22} = -W_{yy}, \quad K_{12} = -W_{xy}; \tag{3}$$

constitutive relations:

$$N_{11} = \frac{Eh}{1-\nu^2}(E_{11} + \nu E_{22}), \quad N_{22} = \frac{Eh}{1-\nu^2}(E_{22} + \nu E_{11}), \quad N_{12} = \frac{Eh}{1+\nu}E_{12},$$
$$M_{11} = D(K_{11} + \nu K_{22}), \quad M_{22} = D(K_{22} + \nu K_{11}), \quad M_{12} = D(1+\nu)K_{12}. \tag{4}$$

Here, $D = Eh^3/[12(1-\nu^2)]$ is the shell's flexural rigidity. Our domain is the rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ and the boundary conditions are

$$U = V = W = W_n = 0. \tag{5}$$

Now, in order to compute the vibration spectrum, we let

$$q_x = q_y = 0, \quad q_z = -\frac{\rho h}{D}W_{tt},$$

where ρ is the constant mass-per-unit-volume of the shell. Then we have a number of choices. For example, we may

1. Substitute relations (3) and (4) into the equilibrium equations (2), resulting essentially in the PDEs (1), then separate out the time variable. This approach turns out to be more complicated than the approach we actually take.
2. Decouple W from the other variables, using either Donnell's equations [13] or the stress function approach [9], resulting in the PDE

$$\Delta^4 W + \frac{Eh}{R^2 D}W_{xxxx} = -\frac{\rho h}{D}\Delta^2 W_{tt}$$

in W alone. *This approach has the serious disadvantage that there are only two W-boundary conditions along each edge, and no apparent way of obtaining two more from the U- and V-boundary conditions.*

3. Introduce the stress function F, resulting in two PDEs in the unknowns W and this stress function, separate out time, and use the U- and V-boundary conditions to derive two boundary conditions in F along each edge. It is this last approach that we choose to take.

To this end, we assume there is a stress potential $F(x, y, t)$ which satisfies

$$F_{yy} = N_{11}, \quad F_{xx} = N_{22}, \quad -F_{xy} = N_{12}.$$

Then it is easy to show that W and F will satisfy the well-known equations

$$\begin{aligned} &\Delta^2 W - \frac{1}{RD} F_{xx} = -\frac{\rho h}{D} W_{tt}, \\ &\Delta^2 F + \frac{Eh}{R} W_{xx} = 0, \qquad 0<x<a, \ 0<y<b. \end{aligned} \tag{6}$$

As for boundary conditions, consider the edge $y = 0$ (the other edges are treated similarly). There we have

$$U(x,0) = V(x,0) = W(x,0) = W_y(x,0) = 0, \quad 0 \leqslant x \leqslant a. \tag{7}$$

So we have two W-boundary conditions along each edge:

$$\begin{aligned} &W(x,0) = W_y(x,0) = W(x,b) = W_y(x,b) = 0, \quad 0 \leqslant x \leqslant a, \\ &W(0,y) = W_x(0,y) = W(a,y) = W_x(a,y) = 0, \quad 0 \leqslant y \leqslant b. \end{aligned} \tag{8}$$

We must now replace the U- and V-conditions with two conditions involving F. First, we note that $U(x,0) = 0$ implies that

$$U_x(x,0) = U_{xx}(x,0) = U_{xxx}(x,0) = \cdots = 0,$$

and, similarly, for V, W and W_y. Now,

$$\begin{aligned} F_{yy} = N_{11} &= \frac{Eh}{1-\nu^2}(E_{11} + \nu E_{22}) = \frac{Eh}{1-\nu^2}\left[U_x + \nu V_y - \frac{\nu}{R} W\right], \\ F_{xx} = N_{22} &= \frac{Eh}{1-\nu^2}(E_{22} + \nu E_{11}) = \frac{Eh}{1-\nu^2}\left[V_y - \frac{1}{R} W + \nu U_x\right] \end{aligned}$$

and

$$F_{xy} = -N_{12} = -\frac{Eh}{1+\nu} E_{12} = -\frac{Eh}{2(1+\nu)}(U_y + V_x).$$

Along $y = 0$, we then have

$$F_{yy} = \frac{Eh\nu}{1-\nu^2} V_y, \quad F_{xx} = \frac{Eh}{1-\nu^2} V_y$$

which gives us our first F-boundary condition:

$$F_{yy}(x,0) - \nu F_{xx}(x,0) = 0, \quad 0 \leqslant x \leqslant a. \tag{9}$$

Further, we have

$$\frac{(1-\nu^2)}{Eh} F_{xxy} = V_{yy} - \frac{1}{R} W_y + \nu U_{xy} = \frac{\nu - 1}{2}(U_{xy} + V_{xx})$$

and

$$\frac{1-\nu^2}{Eh}F_{yyy} = U_{xy} + \nu V_{yy} - \frac{\nu}{R}W_y.$$

Therefore, along $y = 0$,

$$\frac{1-\nu^2}{Eh}F_{xxy} = V_{yy} + \nu U_{xy} = \frac{\nu-1}{2}U_{xy}$$

and

$$\frac{1-\nu^2}{Eh}F_{yyy} = U_{xy} + \nu V_{yy},$$

from which follows our second F-boundary condition

$$F_{yyy}(x,0) + (2+\nu)F_{xxy}(x,0) = 0, \quad 0 \leqslant x \leqslant a. \tag{10}$$

Similarly, we have

$$F_{yy}(x,b) - \nu F_{xx}(x,b) = F_{yyy}(x,b) + (2+\nu)F_{xxy}(x,b) = 0, \quad 0 \leqslant x \leqslant a,$$
$$F_{xx}(0,y) - \nu F_{yy}(0,y) = F_{xxx}(0,y) + (2+\nu)F_{xyy}(0,y) = 0, \quad 0 \leqslant y \leqslant b,$$

and

$$F_{xx}(a,y) - \nu F_{yy}(a,y) = F_{xxx}(a,y) + (2+\nu)F_{xyy}(a,y) = 0, \quad 0 \leqslant y \leqslant b. \tag{11}$$

3. Bolotin's method applied to the problem

Our problem now consists of the two PDEs (6) and the boundary conditions (8)–(11). First, we separate time from the other variables:

$$W(x,y,t) = e^{i\lambda t}w(x,y), \quad F(x,y,t) = e^{i\lambda t}f(x,y).$$

The PDEs (6) then become

$$\Delta^2 w - \frac{1}{RD}f_{xx} = \frac{\rho h}{D}\lambda^2 w,$$
$$\Delta^2 f + \frac{Eh}{R}w_{xx} = 0. \tag{12}$$

The separated boundary conditions are identical to the ones we have, but with $w(x,y)$ replacing $W(x,y,t)$ and $f(x,y)$ replacing $F(x,y,t)$.

Now, Bolotin's asymptotic method [9] entails finding solutions of the form $\sin k_1(x-\xi)\cdot \sin k_2(y-\xi_2)$ on the interior of the domain, then, since these solutions generally do not satisfy the boundary conditions, of finding certain "corrective solutions" near each of the edges and, finally, of "connecting" these solutions.

To this end, we let

$$w(x,y) = \sin k_1(x-\xi_1)\sin k_2(y-\xi_2)$$

and

$$f(x, y) = A \sin k_1(x - \xi_1) \sin k_2(y - \xi_2)$$

for constants k_1, k_2, ξ_1, ξ_2 and A. Then, substituting these into PDEs (12), we arrive at the following relation between the eigenfrequencies λ and the wave numbers k_1 and k_2:

$$\left(\sqrt{\frac{\rho h}{D}}\lambda\right)^2 = (k_1^2 + k_2^2)^2 + \frac{c^4 k_1^4}{(k_1^2 + k_2^2)^2}, \tag{13}$$

where $c^4 = Eh/(R^2 D)$.

Next, we find the corrective solutions of the form $e^{ry} \sin k_1(x - \xi_1)$ near the boundary $y = 0$ (and, similarly, near $y = b$). Specifically, we find those constant values of r for which

$$w(x, y) = e^{ry} \sin k_1(x - \xi_1)$$

and

$$f(x, y) = B_r e^{ry} \sin k_1(x - \xi_1) \tag{14}$$

satisfy the PDEs (12), with the intention of finding linear combinations of these solutions which satisfy the boundary conditions at $y = 0$. Here, the constant B_r may be different for different values of r.

Substituting Eq. (14) into Eq. (12) leads to the 8th degree polynomial equation, given in Ref. [9] (and with typos corrected)

$$(r^2 + k_2^2)[r^2 - (2k_1^2 + k_1^2)]\left[r^4 - 2k_1^2 r^2 + k_1^4 - \frac{c^4 k_1^4}{(k_1^2 + k_2^2)^2}\right] = 0. \tag{15}$$

Therefore, we have the five physically admissible roots

$$r = \pm i k_2, \quad -\sqrt{2k_1^2 + k_2^2}, \quad -k_1\sqrt{1 \pm \frac{c^2}{k_1^2 + k_2^2}}. \tag{16}$$

Here, we have made the further assumption that $1 - c^2/(k_1^2 + k_2^2) > 0$, about which we shall say more in Section 5.

So the corrective solution near the edge $y = 0$ is of the form

$$w = \left[C_{01} \sin k_2 y + C_{02} \cos k_2 y + C_{03} e^{-\sqrt{2k_1^2 + k_2^2}\, y} + C_{04} e^{-k_1\sqrt{1 + (c^2/(k_1^2 + k_2^2))}\, y} + C_{05} e^{-k_1\sqrt{1 - (c^2/(k_1^2 + k_2^2))}\, y}\right] \sin k_1(x - \xi_1),$$

$$f = \left[D_{01} \sin k_2 y + D_{02} \cos k_2 y + D_{03} e^{-\sqrt{2k_1^2 + k_2^2}\, y} + D_{04} e^{-k_1\sqrt{1 + (c^2/(k_1^2 + k_2^2))}\, y} + D_{05} e^{-k_1\sqrt{1 - (c^2/(k_1^2 + k_2^2))}\, y}\right] \sin k_1(x - \xi_1). \tag{17}$$

(To get the corrective solution near $y = b$, we use constants C_{bi} and D_{bi}, and we replace y everywhere by $b - y$).

Now, we apply the boundary conditions at $y = 0$ to these solutions (17), resulting in

$$C_{02} + C_{03} + C_{04} + C_{05} = 0,$$

$$k_2 C_{01} - \sqrt{2k_1^2 + k_2^2}C_{03} - k_1\sqrt{1 + \frac{c^2}{k_1^2 + k_2^2}}\,C_{04} - k_1\sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\,C_{05} = 0,$$

$$(vk_1^2 - k_2^2)D_{02} + [(2 + v)k_1^2 + k_2^2]D_{03} + k_1^2\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right)D_{04} + k_1^2\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right)D_{05} = 0,$$

$$k_2[k_2^2 + (2 + v)k_1^2]D_{01} - \sqrt{2k_1^2 + k_2^2}(vk_1^2 - k_2^2)D_{03} - k_1^3\sqrt{1 + \frac{c^2}{k_1^2 + k_2^2}}\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right)D_{04} - k_1^3\sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right)D_{05} = 0.$$

Further, applying the second PDE in Eq. (12) gives us the additional five equations

$$(k_1^2 + k_2^2)^2 D_{01} - \frac{Eh}{R}k_1^2 C_{01} = 0,$$
$$(k_1^2 + k_2^2)^2 D_{02} - \frac{Eh}{R}k_1^2 C_{02} = 0,$$
$$(k_1^2 + k_2^2)^2 D_{03} - \frac{Eh}{R}k_1^2 C_{03} = 0,$$
$$\frac{c^4 k_1^4}{(k_1^2 + k_2^2)^2}D_{04} - \frac{Eh}{R}k_1^2 C_{04} = 0,$$
$$\frac{c^4 k_1^4}{(k_1^2 + k_2^2)^2}D_{05} - \frac{Eh}{R}k_1^2 C_{05} = 0.$$

We may eliminate all but two of the variables and, after a lengthy computation, we arrive at

$$\alpha C_{01} + \beta C_{02} = 0$$

and, similarly,

$$\alpha C_{b1} + \beta C_{b2} = 0.$$

Here, we have

$$\alpha = a_3c_1d_2 - a_3c_2d_1 + a_2c_3d_1 - a_1c_3d_2 + a_1c_2d_3 - a_2c_1d_3,$$
$$\beta = b_3c_1d_2 - b_3c_2d_1 - b_1c_3d_2 + b_1c_2d_2, \tag{18}$$

with

$$a_1 = c^4k_1k_2[k_2^2 + (2 + v)k_1^2],$$

$$b_1 = \sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right)(k_1^2 + k_2^2)^4,$$

$$c_1 = b_1 + c^4 k_1 \sqrt{2k_1^2 + k_2^2}(k_2^2 - vk_1^2),$$

$$d_1 = b_1 - \sqrt{1 + \frac{c^2}{k_1^2 + k_2^2}}\left(1 + v - \frac{c^2}{k_1^2 - k_2^2}\right)(k_1^2 + k_2^2)^4,$$

$$a_2 = k_2\left[\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right)(k_1^2 + k_2^2)^4 - c^4 k_1^2(k_2^2 + (2 + v)k_1^2)\right],$$

$$c_2 = \sqrt{2k_1^2 + k_2^2}\left[c^4 k_1^2(vk_1^2 - k_2^2) - \left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right)(k_1^2 + k_2^2)^4\right],$$

$$d_2 = -2c^2 k_1 (k_1^2 + k_2^2)^3 \sqrt{1 + \frac{c^2}{k_1^2 + k_2^2}},$$

$$a_3 = a_1\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right),$$

$$b_3 = c^4 k_1^2(vk_1^2 - k_2^2)\sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right) - c^4 k_1 \sqrt{2k_1^2 + k_2^2}(vk_1^2 - k_2^2)\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right),$$

$$c_3 = c^4 k_1^2[k_2^2 + (2 + v)k_1^2] - \sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right) - c^4 k_1 \sqrt{2k_1^2 + k_2^2}(vk_1^2 - k_2^2)\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right),$$

$$d_3 = \sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right)^2(k_1^2 + k_2^2)^4 - \sqrt{1 + \frac{c^2}{k_1^2 + k_2^2}}\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right)^2(k_1^2 + k_2^2)^4.$$

Finally, Bolotin's idea is to "connect" these two corrective solutions via

$$C_{01} \sin k_2 y + C_{02} \cos k_2 y = C_{b1} \sin k_2(b - y) + C_{b2} \cos k_2(b - y),$$

which leads to

$$k_2 b = \tan^{-1}\frac{2\alpha\beta}{\beta^2-\alpha^2} + m\pi, \quad m = 1, 2, \ldots. \tag{19}$$

Eventually, we will solve this equation simultaneously with the equation resulting from the corrective solutions near $x = 0$ and a for the wave numbers k_1 and k_2.

So, let us now treat the boundary condition $x = 0$ (and, similarly, $x = a$). As before, we find those values for r for which

$$w(x, y) = e^{rx}\sin k_2(y - \xi_2)$$

and

$$f(x, y) = B_r e^{rx}\sin k_2(y - \xi_2)$$

satisfy the PDEs (12). Proceeding as before, we arrive at the following 8th degree polynomial equation in r:

$$(r^2 + k_1^2)\left\{r^6 - (k_1^2 + 4k_2^2)r^4 + k_2^2\left[2k_1^2 + 5k_2^2 + \frac{c^4(2k_1^2 + k_2^2)}{(k_1^2 + k_2^2)^2}\right]r^2 - k_2^4\left[k_1^2 + 2k_2^2 + \frac{c^4 k_1^2}{(k_1^2 + k_2^2)^2}\right]\right\} = 0 \tag{20}$$

or, in Bolotin's [9] notation,

$$(r^2 + k_1^2)\Delta(r^2) = 0.$$

We need, in turn, to find the roots of the cubic equation

$$\Delta(x) = 0$$

for which we turn to Cardan's formula. The roots are found to be

$$x_1 = r + s + \frac{k_1^2 + 4k_2^2}{3},$$

$$x_2 = \omega r + \omega^2 s + \frac{k_1^2 + 4k_2^2}{3},$$

$$x_3 = \omega^2 r + \omega s + \frac{k_1^2 + 4k_2^2}{3} \tag{21}$$

where $\omega = e^{2\pi i/3}$ and r and s are given by

$$\begin{Bmatrix} r \\ s \end{Bmatrix} = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

Here, we have

$$p = -\tfrac{1}{3}(k_1^2 + k_2^2)^2 + \frac{c^4 k_2^2(2k_1^2 + k_2^2)}{(k_1^2 + k_2^2)^2}$$

and

$$q = -\tfrac{2}{27}(k_1^2 + k_2^2)^3 + \tfrac{2}{3}\frac{c^4 k_2^2(k_1^2 + 2k_2^2)}{k_1^2 + k_2^2},$$

and, therefore,

$$\begin{aligned}\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} &= \frac{c^2 k_2^2(k_1^2 + k_2^2)}{3\sqrt{3}}\sqrt{-1 + c^4(-\eta_1^4 + 8\eta_1^2\eta_2^2 + 11\eta_2^4) + c^6\eta_2^2(2\eta_1^2 + \eta_2^2)^3} \\ &= \frac{c^2 k_2^2(k_1^2 + k_2^2)}{3\sqrt{3}}[\sqrt{-1} + f(\eta_1, \eta_2)],\end{aligned}$$

where $\eta_i = k_i/(k_1^2 + k_2^2)$, $i = 1, 2$, and f is a polynomial satisfying $f(\eta_1, \eta_2) = O((\eta_1^2 + \eta_2^2)^2)$.

Then,

$$\begin{aligned}\left\{\begin{matrix} r \\ s \end{matrix}\right\} &= \sqrt[3]{\frac{(k_1^2 + k_2^2)^3}{27} - \frac{1}{3}\frac{c^4 k_2^2(k_1^2 + 2k_2^2)}{k_1^2 + k_2^2} \pm \frac{c^2 k_2^2(k_1^2 + k_2^2)}{3\sqrt{3}}[\sqrt{-1} + f(\eta_1, \eta_2)]} \\ &= \frac{k_1^2 + k_2^2}{3}\sqrt[3]{1 \pm 3\sqrt{3}c^4\eta_2^2 \mathrm{i} + g(\eta_1, \eta_2)},\end{aligned}$$

where the polynomial $g(\eta_1, \eta_2) = O(\eta_2^2(\eta_1^2 + \eta_2^2))$. Therefore,

$$\begin{aligned}\left\{\begin{matrix} r \\ s \end{matrix}\right\} &= \frac{k_1^2 + k_2^2}{3}[1 \mp \sqrt{3}\, c^2\eta_2^2 \mathrm{i} + O(\eta_2^2(\eta_1^2 + \eta_2^2))] \\ &= \frac{k_1^2 + k_2^2}{3} \mp \frac{c^2 \mathrm{i}}{\sqrt{3}}\frac{k_2^2}{k_1^2 + k_2^2} + O\left(\frac{k_2^2}{(k_1^2 + k_2^2)^2}\right).\end{aligned}$$

Finally, neglecting the terms of $O(k_2^2/(k_1^2 + k_2^2)^2)$, the roots of $\Delta(x)$ are, from Eq. (21),

$$x_1 = k_1^2 + 2k_2^2,$$

$$x_2 = k_2^2\left(1 + \frac{c^2}{k_1^2 + k_2^2}\right),$$

$$x_3 = k_2^2\left(1 - \frac{c^2}{k_1^2 + k_2^2}\right)$$

and the five physically admissible roots of Eq. (20) are

$$r = \pm \mathrm{i}k_1, -\sqrt{k_1^2 + 2k_2^2}, -k_2\sqrt{1 \pm \frac{c^2}{k_1^2 + k_2^2}}.$$

Proceeding as above, we arrive at the nine equations

$$C_{02} + C_{03} + C_{04} + C_{05} = 0,$$

$$k_1 C_{01} - \sqrt{k_1^2 + 2k_2^2}\, C_{03} - k_2\sqrt{1 + \frac{c^2}{k_1^2 + k_2^2}}\, C_{04} - k_2\sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\, C_{05} = 0,$$

$$(\nu k_2^2 - k_1^2)D_{02} + [k_1^2 + (2+\nu)k_2^2]D_{03} + k_2^2\left(1 + \nu + \frac{c^2}{k_1^2 + k_2^2}\right)D_{04} + k_2^2\left(1 + \nu - \frac{c^2}{k_1^2 + k_2^2}\right)D_{05} = 0,$$

$$k_1\left[k_1^2 + (2+\nu)k_2^2\right]D_{01} + \sqrt{k_1^2 + 2k_2^2}(k_1^2 - \nu k_2^2)D_{03} - k_2\sqrt[3]{1 + \frac{c^2}{k_1^2 + k_2^2}}\left(1 + \nu - \frac{c^2}{k_1^2 + k_2^2}\right)D_{04} - k_2\sqrt[3]{1 - \frac{c^2}{k_1^2 + k_2^2}}\left(1 + \nu + \frac{c^2}{k_1^2 + k_2^2}\right)D_{05} = 0,$$

$$(k_1^2 + k_2^2)^2 D_{01} - \frac{Eh}{R}k_1^2 C_{01} = 0,$$
$$(k_1^2 + k_2^2)^2 D_{02} - \frac{Eh}{R}k_1^2 C_{02} = 0,$$
$$(k_1^2 + k_2^2)^2 D_{03} + \frac{Eh}{R}(k_1^2 + 2k_2^2)C_{03} = 0,$$
$$\frac{c^4 k_2^4}{(k_1^2 + k_2^2)^2}D_{04} + \frac{Eh}{R}k_2^2\left(1 + \frac{c^2}{k_1^2 + k_2^2}\right)C_{04} = 0,$$
$$\frac{c^4 k_2^4}{(k_1^2 + k_2^2)^2}D_{05} + \frac{Eh}{R}k_2^2\left(1 - \frac{c^2}{k_1^2 + k_2^2}\right)C_{05} = 0.$$

We note that the asymmetry in the x and y directions shows up in the last three equations.

Again, proceeding as above, we reduce this system to an equation relating C_{01} and C_{02}, do the same for C_{a1} and C_{a2} at the edge $x = a$, and connect via

$$C_{01}\sin k_1 x + C_{02}\cos k_1 x = C_{a1}\sin k_1(a - x) + C_{a2}\cos k_1(a - x).$$

After a lengthy computation, we arrive at

$$k_1 a = \tan^{-1}\frac{2\alpha\beta}{\beta^2 - \alpha^2} + n\pi, \quad n = 1, 2, \ldots, \tag{22}$$

where α and β are again given by Eq. (18). However, here we have

$$a_1 = -c^4 k_1^3 k_2^4[k_1^2 + (2+\nu)k_2^2],$$

$$b_1 = k_2^5\left(1 - \frac{c^2}{k_1^2 + k_2^2}\right)^{3/2}\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right)(k_1^2 + k_2^2)^4,$$
$$c_1 = b_1 + c^4 k_2^4 (k_1^2 + 2k_2^2)^{3/2}(k_1^2 - v k_2^2),$$
$$d_1 = b_1 - k_2^5\left(1 + \frac{c^2}{k_1^2 + k_2^2}\right)^{3/2}\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right)(k_1^2 + k_2^2)^4,$$

$$a_2 = k_1\left(1 - \frac{c^2}{k_1^2 + k_2^2}\right)^{3/2}(k_1^2 + k_2^2)^4\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right) + c^4 k_1^3[k_1^2 + (2 + v)k_2^2]\sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}},\ c_2 = -c^4(k_1^2 + 2k_2^2)^{3/2}(k_1^2 - v k_2^2)\sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}} - \sqrt{k_1^2 + 2k_2^2}\left(1 - \frac{c^2}{k_1^2 + k_2^2}\right)^{3/2}(k_1^2 + k_2^2)^4\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right),$$

$$d_2 = k_2(k_1^2 + k_2^2)^4\left[\sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\left(1 + \frac{c^2}{k_1^2 + k_2^2}\right)^{3/2}\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right) - \sqrt{1 + \frac{c^2}{k_1^2 + k_2^2}}\left(1 - \frac{c^2}{k_1^2 + k_2^2}\right)^{3/2}\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right)\right],$$

$$a_3 = c^4 k_1^3\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right)[k_1^2 + (2 + v)k_2^2],$$

$$b_3 = -c^4 k_1^2 k_2 (k_1^2 - v k_2^2)\sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right),$$

$$c_3 = -c^4(k_1^2 + 2k_2^2)\left\{k_2[k_1^2 + (2 + v)k_2^2]\sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right) + (k_1^2 - v k_2^2)\sqrt{k_1^2 + 2k_2^2}\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right)\right\},$$

$$d_3 = k_2(k_1^2 + k_2^2)^4\left(1 + \frac{c^2}{k_1^2 + k_2^2}\right)\left[\left(1 + v - \frac{c^2}{k_1^2 + k_2^2}\right)^2\sqrt{1 + \frac{c^2}{k_1^2 + k_2^2}} - \left(1 + v + \frac{c^2}{k_1^2 + k_2^2}\right)^2\sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}\right].$$

The final step is to solve Eqs. (19) and (22) simultaneously for the wave numbers k_1 and k_2, for all pairs of n and m, then substitute each pair (k_1, k_2) into Eq. (13) to determine the corresponding eigenfrequency.

4. The Legendre-tau spectral method applied to the problem

The Legendre-tau spectral method [17] may be applied to problems with domain $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$, and entails approximating the functions involved with truncated sums of products of Legendre polynomials, as is seen below.

We must transform our eigenvalue problem consisting of the PDEs (12) and the boundary conditions in w and f into a problem on the above domain. To this end, we let

$$\phi(x,y) = w\left(\frac{a}{2}(x+1), \frac{b}{2}(y+1)\right)$$

and

$$\psi(x,y) = f\left(\frac{a}{2}(x+1), \frac{b}{a}(y+1)\right), \quad -1 \leqslant x \leqslant 1, \quad -1 \leqslant y \leqslant 1,$$

in which case our original system transforms to the PDEs

$$\frac{16}{a^4}\phi_{xxxx} + \frac{32}{a^2b^2}\phi_{xxyy} + \frac{16}{b^4}\phi_{yyyy} - \frac{1}{RD}\frac{4}{a^2}\psi_{xx} = \tilde{\lambda}^2\phi, \tag{23}$$

$$\frac{16}{a^4}\psi_{xxxx} + \frac{32}{a^2b^2}\psi_{xxyy} + \frac{16}{b^4}\psi_{yyyy} + \frac{Eh}{R}\frac{4}{a^2}\phi_{xx} = 0, \quad -1 < x < 1, \quad -1 < y < 1, \tag{24}$$

and the boundary conditions

$$\phi(x,-1) = \phi_y(x,-1) = \phi(x,1) = \phi_y(x,1) = 0, \quad -1 \leqslant x \leqslant 1,$$

$$\phi(-1,y) = \phi_x(-1,y) = \phi(1,y) = \phi_x(1,y) = 0, \quad -1 \leqslant y \leqslant 1,$$

$$\begin{aligned} &\frac{1}{b^2}\psi_{yy}(x,-1) - \frac{v}{a^2}\psi_{xx}(x,-1) = \frac{1}{b^3}\psi_{yyy}(x,-1) + \frac{2+v}{a^2b}\psi_{xxy}(x,-1) \\ &\quad = \frac{1}{b^2}\psi_{yy}(x,1) - \frac{v}{a^2}\psi_{xx}(x,1) = \frac{1}{b^3}\psi_{yyy}(x,1) + \frac{2+v}{a^2b}\psi_{xxy}(x,1) = 0, \quad -1 \leqslant x \leqslant 1, \end{aligned}$$

$$\begin{aligned} &\frac{1}{a^2}\psi_{xx}(-1,y) - \frac{v}{b^2}\psi_{yy}(-1,y) = \frac{1}{a^3}\psi_{xxx}(-1,y) + \frac{2+v}{ab^2}\psi_{xyy}(-1,y) \\ &\quad = \frac{1}{a^2}\psi_{xx}(1,y) - \frac{v}{b^2}\psi_{yy}(1,y) = \frac{1}{a^3}\psi_{xxx}(1,y) + \frac{2+v}{ab^2}\psi_{xyy}(1,y) = 0, \quad -1 \leqslant y \leqslant 1. \end{aligned} \tag{25}$$

Here $\tilde{\lambda} = \sqrt{(\rho h/D)}\lambda$.

Now, for the Legendre-tau spectral method, we approximate ϕ and ψ using the functions

$$\phi \approx \phi_{N,M}(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm}P_n(x)P_m(y),$$

$$\psi \approx \psi_{N,M}(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{M}{}' b_{nm}P_n(x)P_m(y), \tag{26}$$

where by $\sum_{n=0}^{N} \sum_{m=0}^{M}{}' C_{nm}$ we mean the sum $\sum_{n=0}^{N} \sum_{m=0}^{M} C_{nm}$ with the terms C_{00}, C_{01}, C_{10} and C_{11} neglected (since the PDEs and boundary conditions involve only second and higher order derivative terms of ψ).

We now proceed as in Ref. [18]. Substituting $\phi_{N,M}$ and $\psi_{N,M}$ into the PDEs (23) and (24) and comparing coefficients of $P_i(x)P_j(y)$, $i = 0, \ldots, N-4$, $j = 0, 1, \ldots, M-4$, leads to the $2(N-3)(M-3)$ equations

$$-\tilde{\lambda}^2 a_{ij} + \frac{16}{a^4} \sum_{\substack{q=i+2 \\ q+i \text{ even}}}^{N-2} \sum_{\substack{p=q+2 \\ p+q \text{ even}}}^{N} \left(i+\tfrac{1}{2}\right)\left(q+\tfrac{1}{2}\right)[q(q+1)-i(i+1)][p(p+1)-q(q+1)]a_{pj}$$
$$+ \frac{16}{a^2b^2} \sum_{\substack{p=i+2 \\ p+i \text{ even}}}^{N} \sum_{\substack{q=j+2 \\ q+j \text{ even}}}^{M} \left(i+\tfrac{1}{2}\right)\left(j+\tfrac{1}{2}\right)$$
$$[p(p+1)-i(i+1)][q(q+1)-j(j+1)]a_{pq} + \frac{16}{b^4} \sum_{\substack{q=j+2 \\ q+j \text{ even}}}^{M-2} \sum_{\substack{p=q+2 \\ p+q \text{ even}}}^{M} \left(j+\tfrac{1}{2}\right)\left(q+\tfrac{1}{2}\right)$$
$$[q(q+1)-j(j+1)][p(p+1)-q(q+1)]a_{ip} - \frac{4}{RDa^2} \sum_{\substack{p=i+2 \\ p+i \text{ even}}}^{N} \left(i+\tfrac{1}{2}\right)[p(p+1)-i(i+1)]b_{pj} = 0$$

and

$$\frac{16}{a^4} \sum_{\substack{q=i+2 \\ q+i \text{ even}}}^{N-2} \sum_{\substack{p=q+2 \\ p+q \text{ even}}}^{N} \left(i+\tfrac{1}{2}\right)\left(q+\tfrac{1}{2}\right)[q(q+1)-i(i+1)][p(p+1)-q(q+1)]b_{pj}$$
$$+ \frac{16}{a^2b^2} \sum_{\substack{p=i+2 \\ p+i \text{ even}}}^{N} \sum_{\substack{q=j+2 \\ q+j \text{ even}}}^{M} \left(i+\tfrac{1}{2}\right)\left(j+\tfrac{1}{2}\right)[p(p+1)-i(i+1)][q(q+1)-j(j+1)]b_{pq}$$
$$+ \frac{16}{b^4} \sum_{\substack{q=j+2 \\ q+j \text{ even}}}^{M-2} \sum_{\substack{p=q+2 \\ p+q \text{ even}}}^{M} \left(j+\tfrac{1}{2}\right)\left(q+\tfrac{1}{2}\right)[q(q+1)-j(j+1)][p(p+1)-q(q+1)]b_{ip}$$
$$+ \frac{4Eh}{Ra^2} \sum_{\substack{p=i+2 \\ p+i \text{ even}}}^{N} \left(i+\tfrac{1}{2}\right)[p(p+1)-i(i+1)]a_{pj} = 0,$$

where (here and below) we use the well-known properties of Legendre polynomials (see, e.g., Ref. [17]).

Next, we substitute $\phi_{N,M}$ and $\psi_{N,M}$ into the boundary conditions (25) and compare coefficients of $P_i(x)$, $i = 0, 1, \ldots, N$, or $P_j(y)$, $j = 0, 1, \ldots, M$, resulting in $8N + 8M + 16$ additional equations. However, it is not hard to show that these equations are not linearly independent and, in fact, that there are eight "dependency relationships" at each corner $(x, y) = (\pm 1, \pm 1)$. Therefore, from these we choose $8N + 8M - 16$ linearly independent equations.

Finally, as we have too many equations, we remove four additional equations (we have chosen to remove the equations from Eq. (24), for $i = N-5,\ N-4,\ j = M-5,\ M-4$, although we have experimented with many different rows and have found the results to be stable).

Now, we have a square system in the unknowns a_{nm} and b_{nm} and the eigenvalue λ, which we can write as

$$(\tilde{\lambda}^2 A + B)v = 0$$

for $[2(N+1)(M+1)-4] \times [2(N+1)(M+1)-4]$ matrices A and B.

5. Computational results

In this final section, let us compute the eigenfrequencies for specific cases. We are restricted by the following assumptions:

$$\frac{a}{R}, \frac{b}{R} < 0.1 \quad \text{(shallowness assumption [16])},$$

$$\frac{h^2}{12R^2} \ll 1 \quad \text{(thinness assumption [5, 13])},$$

$$c^2 = \frac{\sqrt{12(1-v^2)}}{Rh} < k_1^2 + k_2^2. \tag{27}$$

This last assumption, as mentioned earlier, ensures that the root

$$r = -k_i \sqrt{1 - \frac{c^2}{k_1^2 + k_2^2}}$$

is a negative real number, thereby ensuring that we do *not* have, in Bolotin's terminology, an edge effect which *degenerates*. We have been led, in our choice of values for c^2, by our knowledge of the wave numbers of the corresponding flat plate (see, e.g., Refs. [18,19]). We should mention here, though, that there obviously are many situations where a shell is shallow and thin, but where Eq. (27) is violated. Although we do not include such cases in this paper, these degenerate problems are of sufficient importance to warrant further investigation.

Now, the Bolotin calculations in this paper again involve solving simultaneously Eqs. (19) and (22), for the wave numbers k_1 and k_2. To this end, we have used the IMSL subroutine DNEQF [20].

As for the spectral calculations, we have used the IMSL routine DGVCRG, which solves the generalized eigenvalue–eigenvector problem

$$\lambda A v = B v. \tag{28}$$

All computations were performed on the DEC Alpha mainframe at Fairfield University.

In every case, we have taken $v = 0.3$ for the value of Poisson's ratio. In Table 1, we look at the first 30 frequencies for the case $a = b = 1$ and $c^4 = 625$. More specifically, we list the wave-cell numbers n and m, the wave numbers k_1 and k_2 (calculated using Bolotin's method), the Bolotin estimates for the frequencies and, finally, the spectral estimates for the frequencies. For the spectral computations, we have used $N = M = 22$ and, comparing results for $N = M = 20$ and

Table 1
A comparison of the first 30 eigenfrequencies for a 1×1 segment of a shallow cylindrical shell, using Bolotin's method and the Legendre-tau spectral method, including the wave-cell numbers, n and m, and the wave numbers, k_1 and k_2

				Frequency	
n	m	k_1	k_2	Bolotin	Spectral
1	1	4.4263	4.5698	40.5476	42.4722
2	1	7.6674	3.8910	74.0361	74.1276
1	2	3.8648	7.7093	74.3763	76.8294
2	2	7.3583	7.3748	108.5602	110.1033
3	1	10.8970	3.6141	131.8841	132.1857
1	3	3.6092	10.9087	132.0272	134.1390
3	2	10.6913	7.1077	164.8589	165.4702
2	3	7.1021	10.6990	164.9156	166.1317
4	1	14.0788	3.4910	210.4524	210.6038
1	4	3.4896	14.0828	210.5036	211.8771
3	3	10.4788	10.4826	219.7043	220.7867
4	2	13.9392	6.9324	242.3929	242.4160
2	4	6.9305	13.9428	242.4364	244.4315
4	3	13.7729	10.3060	295.9237	296.5930
3	4	10.3041	13.7750	295.9305	296.8665
5	1	17.2407	3.4206	308.9791	309.0493
1	5	3.4200	17.2422	308.9906	310.0026
5	2	17.1425	6.8161	340.3522	340.6740
2	5	6.8149	17.1440	340.3601	341.2982
4	4	13.6161	13.6175	370.8448	372.3674
5	3	17.0133	10.1708	392.9158	392.9889
3	5	10.1700	17.0147	392.9302	394.5137
6	1	20.3935	3.3742	427.3087	427.3723
1	6	3.3740	20.3943	427.3108	428.0507
2	6	6.7333	20.3222	458.3314	458.0217
6	2	20.3216	6.7339	458.3334	459.3059
4	5	13.4822	16.8807	466.7335	467.4201
5	4	16.8800	13.4832	466.7408	467.5568
3	6	10.0676	20.2215	510.2673	510.7043
6	3	20.2213	10.0686	510.2923	511.0504

$c^4 = 625$ and $v = 0.3$.

N= $M = 22$, we have found that the frequencies have converged to at least three decimal places. Also, for the subroutine DNEQF, the error is defined to be

$$ERROR(n,m) = \left[\tan^{-1}\frac{2\alpha_x\beta_x}{\beta_x^2 - \alpha_x^2} + n\pi - k_1 a\right]^2 + \left[\tan^{-1}\frac{2\alpha_y\beta_y}{\beta_y^2 - \alpha_y^2} + m\pi - k_2 b\right]^2,$$

where α_x, β_x and α_y, β_y are the quantities α, β from Eqs. (22) and (19), respectively. For the Bolotin results in Table 1, the greatest error is $ERROR(7,4) = 1.1 \times 10^{-6}$.

Table 2
A comparison of the first 30 eigenfrequencies for a 5×2 segment of a shallow cylindrical shell, using Bolotin's method and the Legendre-tau spectral method, including the wave-cell numbers, n and m, and the wave numbers, k_1 and k_2

				Frequency	
n	m	k_1	k_2	Bolotin	Spectral
1	1	0.7731	2.3904	6.3145	6.8589
2	1	1.4565	2.3096	7.4772	7.7782
3	1	2.1132	2.1839	9.2857	9.5293
4	1	2.7633	2.0582	11.9414	12.2018
5	1	3.4086	1.9600	15.5332	15.7847
1	2	0.6812	3.9183	15.8170	15.7950
2	2	1.3573	3.8824	16.9163	16.8978
3	2	2.0259	3.8285	18.7665	18.7859
6	1	4.0491	1.8906	20.0371	20.2764
4	2	2.6872	3.7665	21.4178	21.4782
5	2	3.3420	3.7042	24.9062	24.9991
7	1	4.6859	1.8414	25.4075	25.6158
6	2	3.9915	3.6465	29.2492	29.3620
1	3	0.6631	5.4917	30.5986	30.6276
8	1	5.3201	1.8053	31.6136	31.7164
2	3	1.3249	5.4721	31.6989	31.8009
3	3	1.9845	5.4416	33.5495	33.5813
7	2	4.6365	3.5956	34.4476	34.5690
4	3	2.6412	5.4037	36.1774	36.2216
9	1	5.9527	1.7778	38.6382	38.7909
5	3	3.2946	5.3619	39.6087	39.6687
8	2	5.2779	3.5516	40.4939	40.6161
6	3	3.9449	5.3192	43.8619	43.9327
10	1	6.5840	1.7562	46.4711	46.7495
9	2	5.9166	3.5140	47.3778	47.4968
7	3	4.5921	5.2775	48.9473	49.0290
1	4	0.6547	7.0647	50.3378	50.3369
2	4	1.3088	7.0525	51.4510	51.4558
3	4	1.9620	7.0332	53.3160	53.3311
8	3	5.2365	5.2382	54.8690	54.9580

$c^4 = 16$ and $\nu = 0.3$.

In Tables 2 and 3 we do the same, but for $a = 5$, $b = 2$ and $c^4 = 16$ for Table 2 and $a = 2$, $b = 5$ and $c^4 = 16$ for Table 3. Each spectral calculation here converges to at least 6 decimal places, comparing $N = M = 20$ and $N = M = 22$. Further, the greatest error in DNEQF is approximately 1.0×10^{-9}.

We see from Tables 1–3 that agreement between the two methods is nearly as good as in the case of the flat plate (see Ref. [18]). Further, this agreement suggests that both methods capture the full range of the spectrum.

Also interesting is a comparison between the 5×2 and the 2×5 cases, where it is seen—more easily with the spectral data than Bolotin's—that the frequencies in the former case are alternatively greater than and less than those of the latter.

Table 3
A comparison of the first 30 eigenfrequencies for a 2×5 segment of a shallow cylindrical shell, using Bolotin's method and the Legendre-tau spectral method, including the wave-cell numbers, n and m, and the wave numbers, k_1 and k_2

				Frequency	
n	m	k_1	k_2	Bolotin	Spectral
1	1	2.3114	0.8016	6.2455	6.9924
1	2	2.2551	1.4766	7.3996	7.5346
1	3	2.1533	2.1263	9.2134	9.1629
1	4	2.0427	2.7714	11.8741	11.8171
1	5	1.9526	3.4134	15.4718	15.4545
2	1	3.9106	0.6821	15.8770	16.2412
2	2	3.8730	1.3589	16.9405	17.2194
2	3	3.8205	2.0278	18.7731	18.9552
1	6	1.8870	4.0519	19.9818	19.9842
2	4	3.7606	2.6889	21.4135	21.5283
2	5	3.7002	3.3435	24.8948	24.9740
1	7	1.8395	4.6876	25.3585	25.3704
2	6	3.6439	3.9926	29.2332	29.2958
3	1	5.4902	0.6632	30.6451	30.8441
1	8	1.8042	5.3212	31.5706	31.5827
3	2	5.4698	1.3252	31.7312	31.9131
3	3	5.4392	1.9849	33.5713	33.7255
2	7	3.5939	4.6373	34.4291	34.4835
3	4	5.4016	2.6416	36.1912	36.3211
1	9	1.7772	5.9533	38.6007	38.6159
3	5	5.3602	3.2951	39.6161	39.7270
2	8	3.5505	5.2786	40.4743	40.5239
3	6	5.3179	3.9453	43.8641	43.9584
1	10	1.7558	6.5845	46.4382	46.6003
2	9	3.5133	5.9171	47.3582	47.4047
3	7	5.2765	4.5925	48.9456	49.0297
4	1	7.0642	0.6547	50.3703	50.4912
4	2	7.0518	1.3089	51.4771	51.5948
4	3	7.0324	1.9621	53.3367	53.4488
3	8	5.2375	5.2368	54.8644	54.9411

$c^4 = 16$ and $\nu = 0.3$.

We remark that, as the curvature of the cylinder $\to 0$ (i.e., as $R \to \infty$), the PDE system (6) becomes

$$\Delta^2 W = -\frac{\rho h}{D} W_{tt}, \quad \Delta^2 F = 0,$$

which, of course, describes the vibration of the Kirchhoff thin plate. In Table 4, we include the Bolotin frequencies for the 1×1 shell for various decreasing values of c^4 and, in the last column, those for the plate. We see that, for each mode, the shell frequency decreases monotonically and approaches that of the plate (as expected, of course). And, of course, as $k_1, k_2 \to \infty$, the

Table 4
A comparison of the first 30 eigenfrequencies for a 1×1 segment of a shallow cylindrical shell, with $\nu = 0.3$, for various values of c^4 approaching 0

$c^4 = 450$	$c^4 = 300$	$c^4 = 200$	$c^4 = 100$	$c^4 = 50$	$c^4 = 10$	$c^4 = 1$	$c^4 = 0.1$	Plate
39.083	37.795	36.918	36.023	35.567	35.194	35.105	35.094	35.092
73.719	73.454	73.279	73.100	73.010	72.925	72.905	72.902	72.902
73.954	73.598	73.365	73.132	73.017	72.931	72.908	72.904	72.902
108.257	107.998	107.824	107.650	107.562	107.490	107.472	107.470	107.470
131.821	131.766	131.728	131.682	131.658	131.635	131.629	131.629	131.629
131.915	131.818	131.755	131.687	131.663	131.641	131.632	131.629	131.629
164.736	164.618	164.545	164.470	164.430	164.396	164.388	164.387	164.387
164.768	164.636	164.557	164.472	164.431	164.398	164.389	164.387	164.387
210.434	210.417	210.404	210.384	210.370	210.363	210.362	210.362	210.362
210.464	210.430	210.407	210.389	210.376	210.368	210.364	210.362	210.362
219.598	219.510	219.449	219.388	219.358	219.332	219.325	219.325	219.325
242.342	242.298	242.268	242.233	242.215	242.200	242.197	242.197	242.197
242.371	242.312	242.274	242.234	242.218	242.203	242.198	242.197	242.197
295.857	295.802	295.772	295.737	295.718	295.702	295.698	295.698	295.698
295.866	295.806	295.773	295.737	295.719	295.703	295.699	295.698	295.698
308.968	308.959	308.949	308.939	308.928	308.929	308.929	308.929	308.929
308.974	308.959	308.953	308.945	308.934	308.933	308.931	308.930	308.929
340.314	340.294	340.280	340.263	340.250	340.245	340.244	340.244	340.244
340.329	340.300	340.282	340.266	340.254	340.248	340.245	340.244	340.244
370.784	370.743	370.716	370.688	370.674	370.662	370.659	370.659	370.658
392.885	392.857	392.838	392.818	392.806	392.798	392.796	392.796	392.796
392.911	392.861	392.839	392.818	392.808	392.799	392.796	392.796	392.796
427.303	427.295	427.290	427.285	427.283	427.280	427.281	427.281	427.280
427.306	427.300	427.296	427.291	427.288	427.283	427.282	427.281	427.280
458.309	458.299	458.290	458.280	458.276	458.270	458.270	458.269	458.269
458.317	458.300	458.292	458.284	458.279	458.272	458.270	458.270	458.269
466.698	466.670	466.652	466.633	466.623	466.615	466.613	466.613	466.613
466.700	466.674	466.652	466.634	466.624	466.616	466.613	466.613	466.613
510.241	510.225	510.214	510.199	510.193	510.189	510.189	510.189	510.189
510.247	510.227	510.215	510.200	510.195	510.190	510.189	510.189	510.189

The last column lists the same frequencies for the 1×1 Kirchhoff thin plate.

frequencies in each column approach those of the plate, as well (from Eq. (13)). (We note that this last statement will not necessarily hold if either of the edges $y =$ constant is free because, in this case, $k_2 = 0$ is a wave number, and, as a result, the right side of Eq. (13) equals $k_1^4 + c^4$.)

Lastly, in Fig. 1 we provide plots of the first eight eigenmodes (including multiplicities) for the function W, for the data which are used in Ref. [5] (in their treatment of inextensional vibrations of cylindrical shells). Specifically, we have $R = 33.0$ mm, $h = 0.155$ mm, $\rho = 3.23 \times 10^3$ kg/m^3, $E = 69.0$ GPa and $\nu = 0.3$, and we have computed the function $\phi_{N,M}$ in Eq. (26) at $41 \times 41 = 1681$ grid points. The coefficients a_{nm} correspond to the first $(N+1)(M+1) = (20+1)(20+1)$ entries of the generalized eigenvector v in Eq. (28), and the graphics were performed using MATLAB. The corresponding frequencies are given in Table 5. Also, we have provided, next to

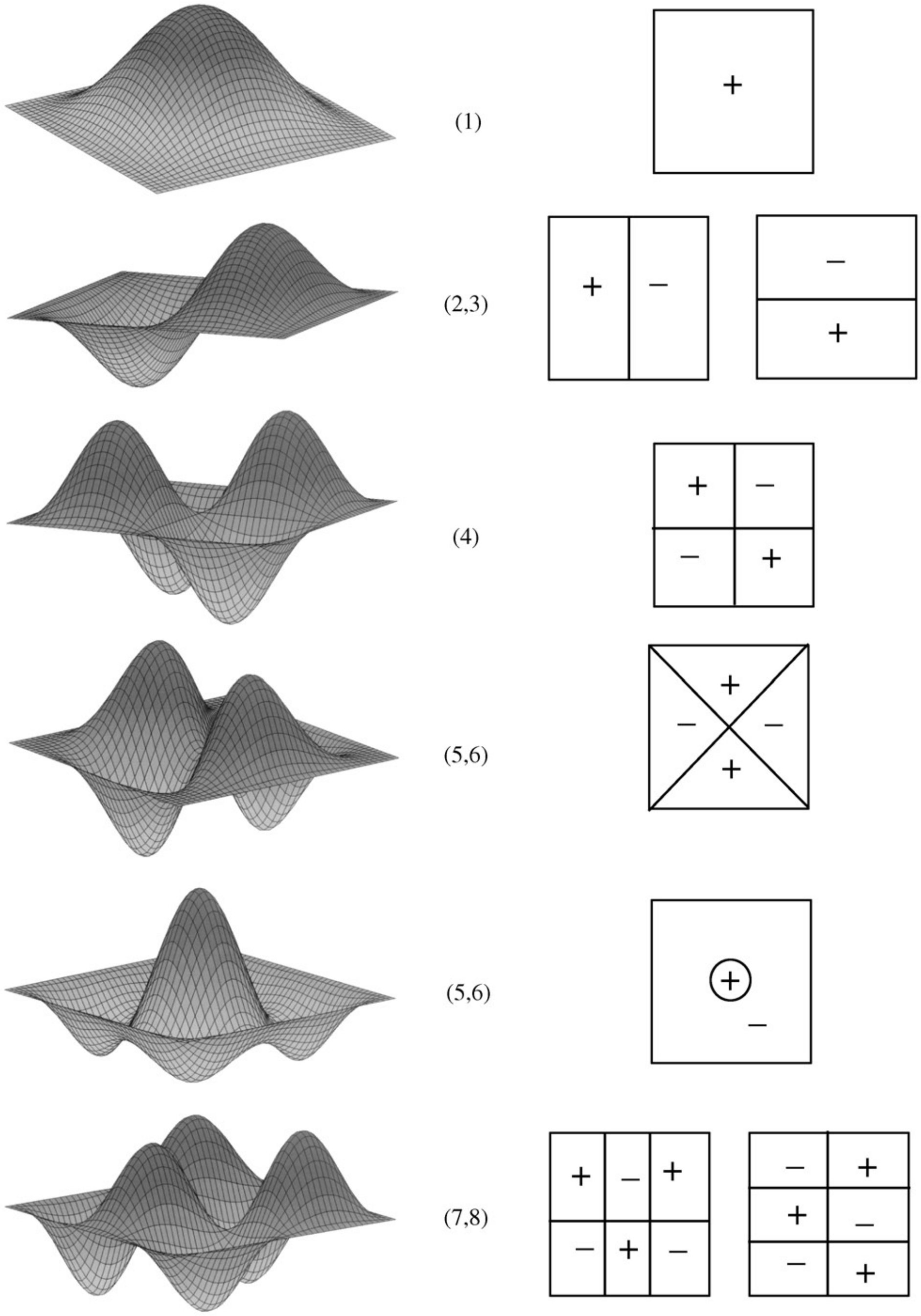

Fig. 1. The first eight eigenmodes, including multiplicities, and corresponding nodal patterns for the function W, using the data given in Ref. [5]: $R = 33.0$ mm, $h = 0.155$ mm, $\rho = 3.23 \times 10^3$ kg/m^3, $E = 69.0$ GPa and $\nu = 0.3$. These modes correspond to the frequencies given in Table 5. Note that, in the nodal figures, "+" represents those regions where $W > 0$; "−", where $W < 0$. The modes are given by the numbers in parentheses.

Table 5
The first eight eigenfrequencies for a 1×1 segment of a shallow cylindrical shell, using Bolotin's method, for the data given in Ref. [5]

n	m	Frequency
1	1	35.098
2	1	72.903
1	2	72.906
2	2	107.470
3	1	131.629
1	3	131.631
3	2	164.387
2	3	164.388

These frequencies correspond to the eigenmodes shown in Fig. 1. Again, n and m are the wave-cell numbers.

each eigenmode, a bird's-eye view of the figure with the nodal lines, and with "+" and "−" representing those regions where $W > 0$ and $W < 0$, respectively.

Although we do not include them here, we plotted the eigenmodes for the Kirchhoff thin plate and have found, not surprisingly, that, at the resolution we have used, the modes for the plate and those for the shell are virtually indistinguishable.

It is interesting to note in Table 5 (as well as in Tables 1 and 4) that there are numerous "almost paired" frequencies. These pairs, of course, correspond to frequencies of the plate which have multiplicity 2. As a result, the 2nd and 3rd eigenmodes essentially will be reflections of each other through the domain's diagonal; the same is true for the 7th and 8th modes. Why, then, are the 5th and 6th modes so different from each other? The answer is that we can find two different linear combinations of these two modes which will exhibit the same symmetry as do the other paired modes.

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